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The Finite Dam with Discrete Additive Input

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SUMMARY

The transient behaviour of a finite discrete dam with additive input and release at unit rate is derived from that of the analogous infinite dam. The method employed is based on Feller's Theory of Recurrent Events. Various (taboo) first entrance and surpassage times are derived for the infinite dam from which the first entrance times for the finite dam are deduced. These lead immediately to the result mentioned. The approach actually affords the determination of joint distributions involving total input and dry time during relevant time intervals as well. Several intermediate results are of interest in their own right. It would seem possible to find the transient behaviour of the Holdaway dam using an analogous approach to the one given (cf. Moran [2]).

1. Introduction

Consider a dam with content $\mathbf{z}_n$ at time $n=0, 1, 2, \ldots$ The content may have any nonnegative integer value. Variation of the content in time is due to release of water at the rate of one unit per time unit and random input. The input during the interval $(n, n+1)$ is $\mathbf{x}_n$; inputs during disjoint unit time intervals are independent and identically distributed with probability generating function $E\{p^{\mathbf{x}_n}\}=P(p)$. We will be lead to consider the equation $p=sP(pq)$. From Rouché's Theorem this is easily seen to have a unique root $w=w(q, s)$ in the domain $|pq|\leqq 1$ if $|sq|<1$.

As we will be concerned with the recurrent event processes imbedded in the Markov chain $\{\mathbf{z}_n, n=0, 1, \ldots\}$, we define the first entrance time $\mathbf{f}_{xy}$ as the time it takes the content process to evolve from the initial state x to the state y for the first time. Taboo first entrance times are denoted by $\mathbf{f}_{u;xy}$. They are defined similar to $\mathbf{f}_{xy}$ with the condition added that the state u is not entered in the mean time. For the description of the relation between the infinite and finite dam we further require the first surpassage time $\mathbf{p}_{xy}$. This is the time required by the content process to go from the state x (via some state $v\leqq y$ if $x>y$) for the first time to a state exceeding y. Taboo first surpassage times $\mathbf{p}_{u;xy}$ are defined similarly; the taboo state must not be entered in the mean time. The distributions of (taboo) first entrance and surpassage times are often defective.

In the sequel we will make repeated use of integrals in the complex domain and introduce the following conventions. Integrals without limits are defined by

$$\oint g(u)du = \frac{1}{2\pi i}\int_C g(u)du\,,$$

where C is a closed contour encircling the origin and such that $|uq|\leqq 1$. The dummy variable p will always be used if C should not enclose the point w; the dummy variable t refers to a contour C encircling the point $t=1$ but again not enclosing w.

2. The Transient Behaviour of the Infinite Dam

In the present section we slightly extend a result of Yeo [4], in deriving the tetravariate generating function (4), (5) for the dam process $\{\mathbf{z}_n, n=0, 1, 2, \ldots\}$, defined as follows:

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$$
\begin{aligned}
\mathbf{z}_0 &= z \\
\mathbf{z}_{n+1} &= \mathbf{z}_n + \mathbf{x}_n - 1 \quad \text{if} \quad \mathbf{z}_n > 0 \\
&= \mathbf{x}_n \quad \text{if} \quad \mathbf{z}_n = 0 .
\end{aligned}
\tag{1}
$$

That is, the content of the dam at time $n+1$ is one unit less than the sum of content at time n and the input during the intervening interval unless the dam was dry at time n, in which case the content at time $n+1$ equals the input during the preceding interval $(n, n+1)$. In other words, the input during an interval is assumed to be available only at the very end of that time interval; this in contrast to Yeo's model as he assumes this to be at the start of the interval.

Further, we will consider the accumulated inflow $\mathbf{X}(n)$ and the total time $\mathbf{d}(n)$ during which the dam is dry in the interval $(0, n)$; the recurrence relations for these processes are for $n=0, 1, 2, \ldots$

$$
\begin{aligned}
\mathbf{X}(n+1) &= \mathbf{X}(n) + \mathbf{x}_n , \\
\mathbf{d}(n+1) &= \mathbf{d}(n) \quad \text{if} \quad \mathbf{z}_n > 0 , \\
&= \mathbf{d}(n) + 1 \quad \text{if} \quad \mathbf{z}_n = 0 ,
\end{aligned}
\tag{2}
$$

while $\mathbf{x}_0 = \mathbf{X}(0) = \mathbf{d}(0) = 0$. From (1) and (2) we obtain for $n=0, 1, 2, \ldots$ that

$$
\begin{aligned}
E\{p^{\mathbf{z}_{n+1}} q^{\mathbf{X}(n+1)} r^{\mathbf{d}(n+1)} \mid \mathbf{z}_0 = z\} &= \frac{1}{p} E\{p^{\mathbf{z}_n} q^{\mathbf{X}(n)} r^{\mathbf{d}(n)} (\mathbf{z}_n > 0) \mid \mathbf{z}_0 = z\} \, E\{(pq)^{\mathbf{x}_n}\} + \\
&\quad + r E\{q^{\mathbf{X}(n)} r^{\mathbf{d}(n)} (\mathbf{z}_n = 0) \mid \mathbf{z}_0 = z\} \, E\{(pq)^{\mathbf{x}_n}\} ,
\end{aligned}
\tag{3}
$$

where (A) stands for the indicator function of the event A. On definition of

$$
Z_z(p, q, r, s) = \sum_{n=0}^{\infty} E\{p^{\mathbf{z}_n} q^{\mathbf{X}(n)} r^{\mathbf{d}(n)} \mid \mathbf{z}_0 = z\} \, s^n
$$

we have from (3) and $Z_z(p, q, r, 0) = p^z$ that

$$
Z_z(p, q, r, s) = \frac{p^{z+1} - s(1-pr) P(pq) Z_z(0, q, r, s)}{p - sP(pq)} . \tag{4}
$$

The quantity $Z_z(0, q, r, s)$ is easily determined from the fact that $\mathbf{z}_n + n = \mathbf{X}_n + \mathbf{d}(n) + z$, so that $Z_z(p, q, r, s)$ is analytic for $|pq| \leqq 1, |qs| < 1, |r/q| < 1$ and $|pr| \leqq 1, |sr| < 1, |q/r| \leqq 1$. The unique root of $p = sP(pq)$ in this domain $w = w(q, s)$ thus must be a zero of the numerator of (4) and hence

$$
Z_z(0, q, r, s) = \frac{w^z}{1 - rw} . \tag{5}
$$

3. Calculation of the First Entrance Times f_{xy}

The first entrance time $\mathbf{f}_{xy}$ for the infinite dam is defined for nonnegative integer x and y by

$$
\mathbf{f}_{xy} = \min\{n : \mathbf{z}_n = y, n > 0 \mid \mathbf{z}_0 = x\} . \tag{6}
$$

From Feller's theory of recurrent events we have that

$$
\oint \frac{Z_x(p, q, r, s)}{p^{y+1}} \, dp = \frac{E\{s^{\mathbf{f}_{xy}} q^{\mathbf{X}(\mathbf{f}_{xy})} r^{\mathbf{d}(\mathbf{f}_{xy})}\}}{1 - E\{s^{\mathbf{f}_{yy}} q^{\mathbf{X}(\mathbf{f}_{yy})} r^{\mathbf{d}(\mathbf{f}_{yy})}\}} , \qquad x \neq y , \tag{7}
$$

and

$$
E\{s^{\mathbf{f}_{xx}} q^{\mathbf{X}(\mathbf{f}_{xx})} r^{\mathbf{d}(\mathbf{f}_{xx})}\} = 1 - \left\{ \oint \frac{Z_x(p, q, r, s)}{p^{x+1}} \, dp \right\}^{-1} = 1 - \left\{ \frac{sw^x}{1 - rw} \oint \frac{(1-pr) P(pq)}{(sP(pq) - p) p^{x+1}} \, dp \right\} , \tag{8}
$$

so that for $x < y$ we have that

$$
E\{s^{\mathbf{f}_{xy}} q^{\mathbf{X}(\mathbf{f}_{xy})} r^{\mathbf{d}(\mathbf{f}_{xy})}\} = w^{x-y} - \frac{(1-rw) w^{-y}}{s} \oint \frac{p^{x-y}}{sP(pq) - p} \, dp \Big/ \oint \frac{(1-pr) P(pq)}{(sP(pq) - p) p^{y+1}} \, dp , \tag{9}
$$

and

$$E\{s^{\mathbf{f}_{yx}}q^{\mathbf{X}(\mathbf{f}_{yx})}r^{\mathbf{d}(\mathbf{f}_{yx})}\} = w^{y-x}\,. \tag{10}$$

4. Calculation of the Taboo First Entrance Times $f_{u;xy}$

The taboo first entrance time $\mathbf{f}_{u;xy}$ for the infinite dam is defined for nonnegative integer u, x and y by

$$\mathbf{f}_{u;xy} = \min\{n : \mathbf{z}_n = y,\ \mathbf{z}_m \neq u,\ 0 < m \leqq n \mid \mathbf{z}_0 = x\}\,. \tag{11}$$

For convenience we adopt the convention that $u < x < y$ and find from the fact that the process $\mathbf{z}_n$ is space homogenous that

$$\begin{aligned} E\{s^{\mathbf{f}_{u;xy}}q^{\mathbf{X}(\mathbf{f}_{u;xy})}r^{\mathbf{d}(\mathbf{f}_{u;xy})}\} &= E\{s^{\mathbf{f}_{0;x-u,y-u}}q^{\mathbf{X}(\mathbf{f}_{0;x-u,y-u})}\} = \\ &= E\{s^{\mathbf{f}_{x-u;y-u}}q^{\mathbf{X}(\mathbf{f}_{x-u;y-u})}(\mathbf{d}(\mathbf{f}_{x-u,y-u})=0)\} = \\ &= w^{x-y} - \frac{w^{u-y}}{s}\oint\frac{p^{x-y}}{sP(pq)-p}\,dp \Big/ \oint\frac{P(pq)}{(sP(pq)-p)p^{y-u+1}}\,dp = \\ &= w^{x-y} - w^{u-y}\oint\frac{p^{x-y}}{sP(pq)-p}\,dp \Big/ \oint\frac{p^{u-y}}{sP(pq)-p}\,dp\,, \end{aligned} \tag{12}$$

and similarly that

$$E\{s^{\mathbf{f}_{x;yy}}q^{\mathbf{X}(\mathbf{f}_{x;yy})}r^{\mathbf{d}(\mathbf{f}_{x;yy})}\} = 1 - w^{x-y} \Big/ \oint\frac{p^{x-y}}{sP(pq)-p}\,dp\,. \tag{13}$$

For the remaining taboo first entrance times we use the familiar argument (cf. Chung [1]) leading to relations like

$$E\{s^{\mathbf{f}_{uy}}\} = E\{s^{\mathbf{f}_{x;uy}}\} + E\{s^{\mathbf{f}_{y;ux}}\}E\{s^{\mathbf{f}_{xy}}\}\,. \tag{14}$$

It follows that

$$\begin{aligned} E\{s^{\mathbf{f}_{y;ux}}q^{\mathbf{X}(\mathbf{f}_{y;ux})}r^{\mathbf{d}(\mathbf{f}_{y;ux})}\} &= \oint\frac{p^{u-y}}{sP(pq)-p}\,dp \Big/ \oint\frac{p^{x-y}}{sP(pq)-p}\,dp + \\ &- \oint\frac{p^{u-x}}{sP(pq)-p}\,dp\oint\frac{(1-pr)P(pq)}{(sP(pq)-p)p^{y+1}}\,dp \Big/ \oint\frac{p^{x-y}}{sP(pq)-p}\,dp\oint\frac{(1-pr)P(pq)}{(sP(pq)-p)p^{x+1}}\,dp \end{aligned} \tag{15}$$

$$\begin{aligned} &E\{s^{\mathbf{f}_{x;uy}}q^{\mathbf{X}(\mathbf{f}_{x;uy})}r^{\mathbf{d}(\mathbf{f}_{x;uy})}\} = \\ &= w^{u-y} - \frac{(1-rw)w^{-y}}{s}\oint\frac{p^{u-x}}{sP(pq)-p}\,dp \Big/ \oint\frac{(1-pr)P(pq)}{(sP(pq)-p)p^{x+1}}\,dp + \\ &- w^{x-y}\oint\frac{p^{u-y}}{sP(pq)-p}\,dp \Big/ \oint\frac{p^{x-y}}{sP(pq)-p}\,dp + \\ &+ w^{x-y}\oint\frac{p^{u-x}}{sP(pq)-p}\,dp\oint\frac{(1-pr)P(pq)}{(sP(pq)-p)p^{y+1}}\,dp \Big/ \oint\frac{p^{x-y}}{sP(pq)-p}\,dp\oint\frac{(1-pr)P(pq)}{(sP(pq)-p)p^{x+1}}\,dp\,, \end{aligned} \tag{16}$$

$$\begin{aligned} &E\{s^{\mathbf{f}_{y;xx}}q^{\mathbf{X}(\mathbf{f}_{y;xx})}r^{\mathbf{X}(\mathbf{f}_{y;xx})}\} = \\ &= 1 - \oint\frac{(1-pr)P(pq)}{(sP(pq)-p)p^{y+1}}\,dp \Big/ \oint\frac{p^{x-y}}{sP(pq)-p}\,dp\oint\frac{(1-pr)P(pq)}{(sP(pq)-p)p^{x+1}}\,dp\,, \end{aligned} \tag{17}$$

$$E\{s^{\mathbf{f}_{x;xy}}q^{\mathbf{X}(\mathbf{f}_{x;xy})}r^{\mathbf{d}(\mathbf{f}_{x;xy})}\}=$$

$$=w^{x-y}\oint\frac{(1-pr)P(pq)}{(sP(pq)-p)p^{y+1}}\,dp\Big/\oint\frac{p^{x-y}}{sP(pq)-p}\,dp\oint\frac{(1-pr)P(pq)}{(sP(pq)-p)p^{p+1}}\,dp\,+$$

$$-(1-rw)w^{-y}\Big/s\oint\frac{(1-pr)P(pq)}{(sP(pq)-p)p^{x+1}}\,dp\,, \tag{18}$$

$$E\{s^{\mathbf{f}_{y;xu}}q^{\mathbf{X}(\mathbf{f}_{y;xu})}r^{\mathbf{d}(\mathbf{f}_{y;xu})}\}=\oint\frac{p^{x-y}}{sP(pq)-p}\,dp\Big/\oint\frac{p^{u-y}}{sP(pq)-p}\,dp\,, \tag{19}$$

and

$$E\{s^{\mathbf{f}_{x;xu}}q^{\mathbf{X}(\mathbf{f}_{x;xu})}r^{\mathbf{d}(\mathbf{f}_{x;xu})}\}=\left(\oint\frac{p^{u-x}}{sP(pq)-p}\,dp\right)^{-1}. \tag{20}$$

5. Calculation of the First Surpassage Times p_{xy}

The first surpassage time $\mathbf{p}_{xy}$ for the infinite dam is defined by

$$\mathbf{p}_{xy}=\min\{n:\mathbf{z}_{n-1}\leqq y<\mathbf{z}_n\,|\,\mathbf{z}_0=x\}\,, \tag{21}$$

where x and y may assume any nonnegative integer value. The only first surpassage times that need be calculated, however, are those for which $x\leqq y$, since it is easily seen that $\mathbf{p}_{yx}=\mathbf{f}_{yx}+\mathbf{p}_{xx}$ for $x<y$, in which the summands are independent.

For the calculation of $\mathbf{p}_{xy}$, $x\leqq y$, we note firstly that the first surpassage time $\mathbf{p}_{xy}$ does not exceed n either if the content process, starting in x, is at time n in a state exceeding y, while the state $y+1$ has not been entered in the mean time, or if the content process, starting in x, has entered the state $y+1$ in the interval $(0, n)$, so that

$$P\{\mathbf{p}_{xy}\leqq n\}=P\{\mathbf{z}_n>y|\mathbf{z}_0=x\}-\sum_{m=0}^{n-1}P\{\mathbf{z}_{n-m}>y|\mathbf{z}_0=y+1\}P\{\mathbf{f}_{x,y+1}=m\}+P\{\mathbf{f}_{x,y+1}<n\}=$$

$$=P\{\mathbf{z}_n>y|\mathbf{z}_0=x\}-\sum_{m=0}^{n}P\{\mathbf{z}_{n-m}>y|\mathbf{z}_0=y+1\}P\{\mathbf{f}_{x,y+1}=m\}+P\{\mathbf{f}_{x,y+1}\leqq n\}\,. \tag{22}$$

In order to obtain the joint distribution of $\mathbf{p}_{xy}$, $\mathbf{X}(\mathbf{p}_{xy})$ and $\mathbf{d}(\mathbf{p}_{xy})$ we note secondly that $\mathbf{d}(\mathbf{p}_{xy})=\mathbf{d}(\mathbf{f}_{x,y+1})$, since there is a one-to-one correspondence between relevant overlapping first entrance and first surpassage times, while during the (possibly infinite) time interval from elapse of the first surpassage time $\mathbf{p}_{xy}$ to elapse of its overlapping first entrance time $\mathbf{f}_{x,y+1}$, the $\mathbf{d}(.)$ process is constant. Hence, when expanding (22), we will consider the input process throughout the interval $(0, n)$, but restrict our attention to the $\mathbf{d}(.)$ process to $(0, \mathbf{f}_{x,y+1})$ or $(0, \mathbf{p}_{xy})$, whichever is more convenient. It follows that

$$\sum_{m=0}^{n}E\{q^{\mathbf{X}(n-m)}\}E\{(\mathbf{p}_{xy}=m)q^{\mathbf{X}(m)}r^{\mathbf{d}(m)}\}=E\{(\mathbf{z}_n>y)q^{\mathbf{X}(n)}r^{\mathbf{d}(n)}|\mathbf{z}_0=x\}\,+$$

$$-\sum_{m=0}^{n}E\{(\mathbf{z}_{n-m}>y)q^{\mathbf{X}(n-m)}r^{\mathbf{d}(n-m)}|\mathbf{z}_0=y+1\}E\{(\mathbf{f}_{x,y+1}=m)q^{\mathbf{X}(m)}r^{\mathbf{d}(m)}\}\,+$$

$$+\sum_{m=0}^{n}E\{q^{\mathbf{X}(n-m)}\}E\{(\mathbf{f}_{x,y+1}=m)q^{\mathbf{X}(m)}r^{\mathbf{d}(m)}\}\,. \tag{23}$$

On taking generating functions with respect to n it is seen that

$$\frac{1}{1-sP(q)}E\{s^{\mathbf{p}_{xy}}q^{\mathbf{X}(\mathbf{p}_{xy})}r^{\mathbf{d}(\mathbf{p}_{xy})}\}=Z_x(1,q,r,s)-\oint\frac{Z_x(p,q,r,s)}{(1-p)p^{y+1}}dp+$$

$$-\left\{Z_{y+1}(1,q,r,s)-\oint\frac{Z_{y+1}(p,q,r,s)}{(1-p)p^{y+1}}dp\right\}E\{s^{\mathbf{f}_{x,y+1}}q^{\mathbf{X}(\mathbf{f}_{x,y+1})}r^{\mathbf{d}(\mathbf{f}_{x,y+1})}\}+$$

$$+\frac{1}{1-sP(q)}E\{s^{\mathbf{f}_{x,y+1}}q^{\mathbf{X}(\mathbf{f}_{x,y+1})}r^{\mathbf{d}(\mathbf{f}_{x,y+1})}\}=\oint\frac{t^{x-y}}{(1-t)(sP(tq)-t)}dt+$$

$$-\frac{s}{1-rw}\oint\frac{(1-tr)P(tq)}{(1-t)(sP(tq)-t)t^{y+1}}dt\{w^x-w^{y+1}E\{s^{\mathbf{f}_{x,y+1}}q^{\mathbf{X}(\mathbf{f}_{x,y+1})}r^{\mathbf{d}(\mathbf{f}_{x,y+1})}\}\},\tag{24}$$

where the last two integrals are to be taken along a circular contour in the t-plane such that $|tq|<1$ and that the contour encircles the point $t=1$ but not the point w, the latter condition may require a local indentation of the otherwise circular contour.

Thus we obtain from (9) that

$$E\{s^{\mathbf{p}_{xy}}q^{\mathbf{X}(\mathbf{p}_{xy})}r^{\mathbf{d}(\mathbf{p}_{xy})}\}=$$

$$=(1-sP(q))\left\{\oint\frac{t^{x-y}}{(1-t)(sP(tq)-t)}dt+\right.$$

$$\left.-\oint\frac{p^{x-y-1}}{sP(pq)-p}dp\oint\frac{(1-tr)P(tq)}{(1-t)(sP(tq)-t)t^{y+1}}dt\Big/\oint\frac{(1-pr)P(pq)}{(sP(pq)-p)p^{y+2}}dp\right\},\tag{25}$$

in particular

$$E\{s^{\mathbf{p}_{xx}}q^{\mathbf{X}(\mathbf{p}_{xx})}r^{\mathbf{d}(\mathbf{p}_{xx})}\}=$$

$$=1-\frac{1-sP(q)}{sP(0)}\oint\frac{(1-tr)P(tq)}{(1-t)(sP(tq)-t)t^{x+1}}dt\Big/\oint\frac{(1-pr)P(pq)}{(sP(pq)-p)p^{x+2}}dp.\tag{26}$$

Finally, we reïnforce a previous remark by noting that the trivariate generating function of $\mathbf{p}_{yx}$, $\mathbf{X}(\mathbf{p}_{yx})$ and $\mathbf{d}(\mathbf{p}_{yx})$ is the product of (10) and (26).

6. Calculation of the Taboo First Surpassage Times $p_{u;xy}$

The taboo first surpassage time $\mathbf{p}_{u;xy}$ for the infinite dam is defined by

$$\mathbf{p}_{u;xy}=\min\{n:\mathbf{z}_{n-1}\leqq y<\mathbf{z}_n,\ \mathbf{z}_m\neq u,\ 0<m<n|\mathbf{z}_0=x\},\tag{27}$$

where u, x and y may have any nonnegative integer values. Only the following nine cases are of interest however $(u<x<y)$.

$\mathbf{p}_{u;xy}$	$\mathbf{p}_{u;yx}$	$\mathbf{p}_{u;xx}$
$\mathbf{p}_{x;xy}$	$\mathbf{p}_{y;xu}$	$\mathbf{p}_{y;xy}$
$\mathbf{p}_{x;uy}$	$\mathbf{p}_{y;yx}$	$\mathbf{p}_{x;xx}$

The first three are obtained easiest by translation, such that the taboo state becomes the zero state, thus we have that

$$E\{s^{\mathbf{p}_{u;xy}}q^{\mathbf{X}(\mathbf{p}_{u;xy})}r^{\mathbf{d}(\mathbf{p}_{u;xy})}\}=$$

$$=(1-sP(q))\left\{\oint\frac{t^{x-y}}{(1-t)(sP(tq)-t)}dt+\right.$$

$$\left.-\oint\frac{p^{x-y-1}}{sP(pq)-p}dp\oint\frac{t^{u-y}}{(1-t)(sP(tq)-t)}dt\Big/\oint\frac{p^{u-y-1}}{sP(pq)-p}dp\right\},\tag{28}$$

$$E\{s^{\mathbf{p}_{u;xx}} q^{\mathbf{X}(\mathbf{p}_{u;xx})} r^{\mathbf{d}(\mathbf{p}_{u;xx})}\} = $$
$$= 1 - \frac{1-sP(q)}{sP(0)} \oint \frac{t^{u-x}}{(1-t)(sP(tq)-t)}\, dt \Big/ \oint \frac{p^{u-x-1}}{sP(pq)-p}\, dp\,, \tag{29}$$

while the analogous generating function for $\mathbf{p}_{u;yx}$ is the product of (10) and (29). The remaining six taboo first surpassage times are found in the same spirit as the taboo first entrance times from relations like

$$E\{s^{\mathbf{p}_{uy}}\} = E\{s^{\mathbf{p}_{x;uy}}\} + E\{s^{\mathbf{f}_{y+1;ux}}\}\, E\{s^{\mathbf{p}_{xy}}\}\,. \tag{30}$$

Thus we have that

$$E\{s^{\mathbf{p}_{x;xy}} q^{\mathbf{X}(\mathbf{p}_{x;xy})} r^{\mathbf{d}(\mathbf{p}_{x;xy})}\} = (1-sP(q)) \times$$
$$\times \left\{ \oint \frac{t^{x-y}}{(1-t)(sP(tq)-t)}\, dt \oint \frac{(1-pr)P(pq)}{(sP(pq)-p)p^{y+2}}\, dp \Big/ \oint \frac{p^{x-y-1}}{sP(pq)-p}\, dp \oint \frac{(1-pr)P(pq)}{(sP(pq)-p)p^{x+1}}\, dp + \right.$$
$$\left. - \oint \frac{(1-tr)P(tq)}{(1-t)(sP(tq)-t)t^{y+1}}\, dt \Big/ \oint \frac{(1-pr)P(pq)}{(sP(pq)-p)p^{x+1}}\, dp \right\}, \tag{31}$$

$$E\{s^{\mathbf{p}_{y;xu}} q^{\mathbf{X}(\mathbf{p}_{y;xu})} r^{\mathbf{d}(\mathbf{p}_{y;xu})}\} = \left\{ \oint \frac{p^{x-y}}{sP(pq)-p}\, dp \Big/ \oint \frac{p^{u-y}}{sP(pq)-p}\, dp \right\} \times$$
$$\times \left\{ 1 - \frac{1-sP(q)}{sP(0)} \oint \frac{(1-tr)P(tq)}{(1-t)(sP(tq)-t)t^{u+1}}\, dt \Big/ \oint \frac{(1-pr)P(pq)}{(sP(pq)-p)p^{u+2}}\, dp \right\}, \tag{32}$$

$$E\{s^{\mathbf{p}_{y;xy}} q^{\mathbf{X}(\mathbf{p}_{y;xy})} r^{\mathbf{d}(\mathbf{p}_{y;xy})}\} =$$
$$= (1-sP(q)) \left\{ \oint \frac{t^{x-y}}{(1-t)(sP(tq)-t)}\, dt + \right.$$
$$\left. - \oint \frac{p^{x-y}}{sP(pq)-p}\, dp \oint \frac{(1-tr)P(tq)}{(1-t)(sP(tq)-t)t^{y+1}}\, dt \Big/ \oint \frac{(1-pr)P(pq)}{(sP(pq)-p)p^{y+1}}\, dp \right\} +$$
$$- sP(0) \left\{ \oint \frac{p^{x-y-1}}{sP(pq)-p}\, dp - \oint \frac{p^{x-y}}{sP(pq)-p}\, dp \oint \frac{(1-pr)P(pq)}{(sP(pq)-p)p^{y+2}}\, dp \Big/ \oint \frac{(1-pr)P(pq)}{(sP(pq)-p)p^{y+1}}\, dp \right\}, \tag{33}$$

$$E\{s^{\mathbf{p}_{x;uy}} q^{\mathbf{X}(\mathbf{p}_{x;uy})} r^{\mathbf{d}(\mathbf{p}_{x;uy})}\} =$$
$$= (1-sP(q)) \left\{ \oint \frac{t^{u-y}}{(1-t)(sP(tq)-t)}\, dt + \right.$$
$$\left. - \oint \frac{t^{x-y}}{(1-t)(sP(tq)-t)}\, dt \oint \frac{p^{u-y-1}}{sP(pq)-p}\, dp \Big/ \oint \frac{p^{x-y-1}}{sP(pq)-p}\, dp \right\} +$$
$$- (1-sP(q)) \oint \frac{p^{u-x}}{sP(pq)-p}\, dp \Big/ \oint \frac{(1-pr)P(pq)}{(sP(pq)-p)p^{x+1}}\, dp \left\{ \oint \frac{(1-tr)P(tq)}{(1-t)(sP(tq)-t)t^{y+1}}\, dt + \right.$$
$$\left. - \oint \frac{t^{x-y}}{(1-t)(sP(tq)-t)}\, dt \oint \frac{(1-pr)P(pq)}{(sP(pq)-p)p^{y+2}}\, dp \Big/ \oint \frac{p^{x-y-1}}{sP(pq)-p}\, dp \right\}, \tag{34}$$

$$E\{s^{\mathbf{p}_{y;yx}} q^{\mathbf{X}(\mathbf{p}_{y;yx})} r^{\mathbf{d}(\mathbf{p}_{y;yx})}\} =$$
$$= \left(\oint \frac{p^{x-y}}{sP(pq)-p}\, dp \right)^{-1} \left\{ 1 - \frac{1-sP(q)}{sP(0)} \oint \frac{(1-tr)P(tq)}{(1-t)(sP(tq)-t)t^{x+1}}\, dt \cdot \Big/ \oint \frac{(1-pr)P(pq)}{(sP(pq)-p)p^{x+2}}\, dp \right\} \tag{35}$$

and

$$E\{s^{\mathbf{p}_{x;xx}}q^{\mathbf{X}(\mathbf{p}_{x;xx})}r^{\mathbf{d}(\mathbf{p}_{x;xx})}\} =$$

$$= sP(0)\oint \frac{(1-pr)P(pq)}{(sP(pq)-p)p^{x+2}}\,dp \Big/ \oint \frac{(1-pr)P(pq)}{(sP(pq)-p)p^{x+1}}\,dp +$$

$$-(1-sP(q))\oint \frac{(1-tr)P(tq)}{(1-t)(sP(tq)-t)t^{x+1}}\,dt \Big/ \oint \frac{(1-pr)P(pq)}{(sP(pq)-p)p^{x+1}}\,dp\,. \tag{36}$$

7. Calculation of the First Entrance Times $f^{\star}_{xy}$

We now turn our attention to the finite dam process $\{\mathbf{z}^*_n,\, n=0, 1, 2, \ldots\}$, defined for integer K as follows

$$\begin{aligned}
\mathbf{z}^*_0 &= z \\
\mathbf{z}^*_{n+1} &= \mathbf{z}^*_n+\mathbf{x}_n-1 && \text{if} \quad \mathbf{z}^*_n>0,\ \mathbf{z}^*_n+\mathbf{x}_n \leqq K+1 \\
&= \mathbf{x}_n && \text{if} \quad \mathbf{z}^*_n=0,\ \mathbf{x}_n \leqq K \\
&= K && \text{if} \quad \mathbf{z}^*_n+\mathbf{x}_n \geqq K+1\,.
\end{aligned} \tag{37}$$

The dry time process $\{\mathbf{d}^*(n),\, n=0, 1, 2, \ldots\}$ is defined analogous to (2). For this finite dam process we define the first entrance time $\mathbf{f}^*_{xy}$ as

$$\mathbf{f}^*_{xy} = \min\{n: \mathbf{z}^*_n = y,\ n>0 \,|\, \mathbf{z}^*_0 = x\}\,, \tag{38}$$

where x and y may have any nonnegative integer values not exceeding K.

In view of (1) and (37) we now establish a relation between the first entrance times for the finite dam and the taboo first entrance and surpassage times for the infinite dam. Consider the exhaustive and mutually exclusive ways in which the finite dam process may develop from an initial state x to a first entrance into the state y; the number of intervening overflows serves as the discriminator

$$E\{s^{\mathbf{f}^*_{xy}}\} = E\{s^{\mathbf{f}_{K+1;xy}}\} + E\{s^{\mathbf{p}_{y;xK}}\} \sum_{n=0}^{\infty} \{E\{s^{\mathbf{p}_{y;KK}}\}\}^n E\{s^{\mathbf{f}_{K+1;Ky}}\} =$$

$$= E\{s^{\mathbf{f}_{K+1;xy}}\} + \frac{E\{s^{\mathbf{p}_{y;xK}}\}\,E\{s^{\mathbf{f}_{K+1;Ky}}\}}{1-E\{s^{\mathbf{p}_{y;KK}}\}}\,. \tag{39}$$

This argument leads for $x<y$ to

$$E\{s^{\mathbf{f}^*_{xy}}q^{\mathbf{X}(\mathbf{f}^*_{xy})}r^{\mathbf{d}^*(\mathbf{f}^*_{xy})}\} = \oint \frac{t^{x-K}}{(1-t)(sP(tq)-t)}\,dt \Big/ \oint \frac{t^{y-K}}{(1-t)(sP(tq)-t)}\,dt +$$

$$-\oint \frac{(1-tr)P(tq)}{(1-t)(sP(tq)-t)t^{K+1}}\,dt \oint \frac{p^{x-y}}{sP(pq)-p}\,dp \Big/ \oint \frac{t^{y-K}}{(1-t)(sP(tq)-t)}\,dt \oint \frac{(1-pr)P(pq)}{(sP(pq)-p)p^{y+1}}\,dp\,, \tag{40}$$

$$E\{s^{\mathbf{f}^*_{yy}}q^{\mathbf{X}(\mathbf{f}^*_{yy})}r^{\mathbf{d}^*(\mathbf{f}^*_{yy})}\} =$$

$$= 1 - \oint \frac{(1-tr)P(tq)}{(1-t)(sP(tq)-t)t^{K+1}}\,dt \Big/ \oint \frac{t^{y-K}}{(1-t)(sP(tq)-t)}\,dt \oint \frac{(1-pr)P(pq)}{(sP(pq)-p)p^{y+1}}\,dp\,, \tag{41}$$

and

$$E\{s^{\mathbf{f}^*_{yx}}q^{\mathbf{X}(\mathbf{f}^*_{yx})}r^{\mathbf{d}^*(\mathbf{f}^*_{yx})}\} = \oint \frac{t^{y-K}}{(1-t)(sP(tq)-t)}\,dt \Big/ \oint \frac{t^{x-K}}{(1-t)(sP(tq)-t)}\,dt\,. \tag{42}$$

8. The Transient Behaviour of the Finite Dam

From the results of the previous section, the transient behaviour of the finite dam is now found, reversing the arguments in section 3. The state of the finite dam process at time n is y if it is so for the first time or second time or third time etc. Hence we have for $x \neq y$ that

$$\sum_{n=0}^{\infty} s^n E\{(\mathbf{z}_n^* = y) q^{\mathbf{X}(n)} r^{\mathbf{d}^*(n)} | \mathbf{z}_0^* = x\} = \frac{E\{s^{\mathbf{f}_{xy}^*} q^{\mathbf{X}(\mathbf{f}_{xy}^*)} r^{\mathbf{d}^*(\mathbf{f}_{xy}^*)}\}}{1 - E\{s^{\mathbf{f}_{yy}^*} q^{\mathbf{X}(\mathbf{f}_{yy}^*)} r^{\mathbf{d}^*(\mathbf{f}_{yy}^*)}\}}, \tag{43}$$

while for $x = y$ the numerator of the right-hand side should read unity. It follows for $x < y$ that

$$\sum_{n=0}^{\infty} s^n E\{(\mathbf{z}_n^* = y) q^{\mathbf{X}(n)} r^{\mathbf{d}^*(n)} | \mathbf{z}_0^* = x\} =$$

$$= \oint \frac{(1-pr) P(pq)}{(sP(pq) - p) p^{y+1}} dp \oint \frac{t^{x-K}}{(1-t)(sP(tq) - t)} dt \Big/ \oint \frac{(1-tr) P(tq)}{(1-t)(sP(tq) - t) t^{K+1}} dt +$$

$$- \oint \frac{p^{x-y}}{sP(pq) - p} dp, \tag{44}$$

and for $x \leq y$ that

$$\sum_{n=0}^{\infty} s^n E\{(\mathbf{z}_n^* = x) q^{\mathbf{X}(n)} r^{\mathbf{d}^*(n)} | \mathbf{z}_0^* = y\} =$$

$$= \oint \frac{(1-pr) P(pq)}{(sP(pq) - p) p^{x+1}} dp \oint \frac{t^{y-K}}{(1-t)(sP(tq) - t)} dt \Big/ \oint \frac{(1-tr) P(tq)}{(1-t)(sP(tq) - t) t^{K+1}} dt. \tag{45}$$

For the tetravariate generating function of content $\mathbf{z}_n^*$, input $\mathbf{X}(n)$ and dry time $\mathbf{d}^*(n)$ we find

$$Z_z^*(p, q, r, s) = \sum_{n=0}^{\infty} s^n E\{p^{\mathbf{z}_n^*} q^{\mathbf{X}(n)} r^{\mathbf{d}^*(n)} | \mathbf{z}_0^* = z\} = \oint \frac{1-(p/u)^{K+1}}{u-p} \frac{1}{u - sP(uq)} \times$$

$$\times \left\{ u^{z+1} + (1-ur) P(uq) \oint \frac{t^{z-K}}{(1-t)(sP(tq) - t)} dt \Big/ \oint \frac{(1-tr) P(tq)}{(1-t)(sP(tq) - t) t^{K+1}} dt \right\} du, \tag{46}$$

where the integral with dummy variable u should be taken along a closed contour C as defined in section 1, enclosing the origin but neither encircling the points $u = w$ nor $u = p$. From (4), (5) and (46) it is seen that if the dam is full initially there is a simple relation with the infinite dam:

$$Z_K^*(p, q, r, s) = -\frac{1}{1 - sP(q)} \oint \frac{1-(p/u)^{K+1}}{u-p} \frac{(1-ur) P(uq)}{u - sP(uq)} du \Big/ \oint \frac{(1-tr) P(tq)}{(1-t)(sP(tq) - t) t^{K+1}} dt =$$

$$= \frac{1}{1 - sP(q)} \frac{\oint \frac{1-(p/u)^{K+1}}{u-p} Z_K(u, q, r, s) du}{Z_K(0, q, 0, s) - Z_K(1, q, r, s) + \oint [(1-v) v^{K+1}]^{-1} Z_K(v, q, r, s) dv}. \tag{47}$$

That is

$$\frac{\sum_{n=0}^{\infty} s^n E\{p^{\mathbf{z}_n^*} q^{\mathbf{X}(n)} r^{\mathbf{d}^*(n)} | \mathbf{z}_0^* = K\}}{\sum_{n=0}^{\infty} s^n E\{q^{\mathbf{X}(n)}\}} =$$

$$= \frac{\sum_{n=0}^{\infty} s^n E\{p^{\mathbf{z}_n} q^{\mathbf{X}(n)} r^{\mathbf{d}(n)} (\mathbf{z}_n \leq K) | \mathbf{z}_0 = K\}}{\sum_{n=0}^{\infty} s^n E\{q^{\mathbf{X}(n)}\} - \sum_{n=0}^{\infty} s^n E\{q^{\mathbf{X}(n)} r^{\mathbf{d}(n)} (\mathbf{z}_n > K) | \mathbf{z}_0 = K\}}. \tag{48}$$

Finally, we note that (46) and also the various intermediate results contain information regarding the overflow process, the accepted input and the output process (cf. Roes [3]).

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